

11-1 Permutations and Combinations

How many different ways can you arrange 3 students in a row?

How many different ways can you arrange 4 letters?

How many different ways can you arrange a math book, an English book, a science book, a Spanish book, and a health book on a shelf?

*These are questions that can be answered using the multiplication counting principle.

Multiplication Counting Principle

You can determine the number of arrangements by multiplying together the number of choices from each position or situation.

Definitions:

Factorial – The product of a given integer and all smaller positive integers.

The factorial of n is written $n!$ and is read aloud “ n factorial”.

$$\text{ie. } 5! = 5 \times 4 \times 3 \times 2 \times 1$$

$$12! = 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

Note: $0! = 1$

Permutation – A selection of objects in which the order of the objects matters.

The choice of r things from a set of n things where the order matters can be found using the Permutation Formula

$${}_n P_r = \frac{n!}{(n-r)!}$$

Combination – A selection of objects in which the order of the objects does not matter.

The choice of r things from a set of n things where the order does not matter can be found using the Combination Formula

$${}_n C_r = \frac{n!}{r!(n-r)!}$$

Example 1: Solve each problem using the multiplication counting principal.

a. How many different 6-letter arrangements can the letters in the word "GRATES" be arranged? Assume that a letter cannot be used more than once.

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

b. How many different 4-letter arrangements can be made from the letters in the word "MATH"? Assume that a letter cannot be used more than once.

$$4 \cdot 3 \cdot 2 \cdot 1 = \boxed{24}$$

c. How many different 5-letter arrangements can be made from the letters in the word "TABLE"? Assume that a letter cannot be used more than once.

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{120}$$

d. Joleen is on a shopping spree. She buys six tops, three shorts, and 4 pairs of sandals. How many different outfits consisting of a top, shorts, and sandals can she create from her new purchases?

$$6 \cdot 3 \cdot 4 = 72$$

Example 2: Find each value.

a. $6! =$

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =$$

$$720$$

b. $(7 - 2)! =$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =$$

$$120$$

c. $\frac{6!}{3!} =$

$$\frac{720}{6} = 120$$

d. $0! = 1$

$$\text{e. } 9! = \boxed{362880}$$

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$\text{f. } (4 + 1)!$$

$$5! =$$

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$\boxed{120}$$

$$\text{g. } 5! \cdot 2!$$

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$120$$

$$2 \cdot 1$$

$$120 \cdot 2$$

$$2$$

$$\boxed{240}$$

$$\text{h. } \frac{3!}{0!} = \frac{3 \cdot 2 \cdot 1}{1} = \frac{6}{1} = \boxed{6}$$

i. How many different ways can you arrange the numbers 1 – 7 in a row, without repeating any number?

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =$$

$$5040$$

Example 3: Find each value.

a. ${}_8P_5$

$$\frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$$

$$\boxed{6720}$$

b. ${}_3P_3$

$$\frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3 \cdot 2 \cdot 1}{1} =$$

$$\boxed{6}$$

c. Find the number of permutations of 10 items taken 3 at a time.

$${}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} =$$

$$\boxed{720}$$

d. Find the number of permutations of 5 items taken 2 at a time.

$${}_5P_2$$

$$\frac{5!}{(5-2)!}$$

$$\frac{5!}{3!}$$

$$\frac{5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{3 \cdot 2 \cdot 1}}$$

$$\boxed{20}$$

Example 4: Find each value.

a. 8C_5

$$\frac{8!}{5!(8-5)!} = \frac{8!}{5!3!} =$$

$$\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \boxed{56}$$

b. 3C_3

$$\frac{3!}{3!(3-3)!} = \frac{3!}{3!0!} = \boxed{1}$$

c. Find the number of combinations of 10 items taken 3 at a time.

$${}^{10}C_3 = \frac{10!}{3!(10-3)!}$$

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$\frac{720}{6} = \boxed{120}$$

d. Find the number of combinations of 5 items taken 2 at a time.

$$\frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{20}{2} = 10$$

Example 5: For each situation, determine whether to use a permutation or combination.

a. How many different teams of 11 players can be chosen from a soccer team of 16?

Combination ${}_{16}C_{11} = \frac{16!}{11!(16-11)!} = \frac{16!}{11!5!} = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{34944}{8}$

4368

b. Ten people apply for the president and vice-president of company ABC. In how many ways can these positions be filled?

permutation ${}_{10}P_2 = \frac{10!}{(10-2)!} = \frac{10!}{8!} = \frac{10 \cdot 9 \cdot 8!}{8!} = 90$

c. Fifteen people apply for positions on a 5-person committee. In how many ways can this committee be formed?

combination ${}_{15}C_5 = \frac{15!}{5!(15-5)!} = \frac{\cancel{15} \cdot \cancel{14} \cdot \cancel{13} \cdot \cancel{12} \cdot \cancel{11} \cdot \cancel{10}!}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot 10!} =$

$$\frac{24024}{8} = \boxed{3003}$$

d. The Classical Studies Department has 7 faculty members. Three must attend the graduation ceremonies. How many different groups of 3 can be chosen?

combination ${}_{7}C_3 = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{\cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4}!}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot 4!} =$

$$\frac{35}{1} = \boxed{35}$$

e. The final night of the Folklore Festival will feature 3 different bands. There are 7 bands to choose from. How many different programs are possible?

permutation

$${}^7P_3 = \frac{7!}{(7-3)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 1210$$

f. There are 500 people competing in a race, how many different possible outcomes are there for first, second, and third place?

permutation

$${}^{500}P_3 = \frac{500 \cdot 499 \cdot 498 \cdot 497!}{497!} = 124,251,000$$

*Some problems cannot be solved using the permutation or combination formulas, but can still be solved using the multiplication counting principle.

Example 6: Solve each problem using permutations, combinations, or the multiplication counting principal.

a. A lottery ticket requires 5 numbers to be chosen from 1 – 50 (repetition of numbers is not allowed). How many possible different lottery tickets can be sold?

$$50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = \boxed{254,251,200}$$

$${}_{50}P_5 = \frac{50!}{(50-5)!} = \frac{50!}{45!} = \frac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot \cancel{45!}}{\cancel{45!}}$$

b. A lottery ticket requires 5 numbers to be chosen from 1 – 50 (repetition of numbers is allowed). How many possible different lottery tickets can be sold?

$$50 \cdot 50 \cdot 50 \cdot 50 \cdot 50 = 312,500,000$$

c. Ten students are in a race. First, second, and third places will win medals. In how many ways can 10 runners finish first, second, and third (no ties allowed)?

$$10 \cdot 9 \cdot 8 = \boxed{720}$$